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Embedding a Differentiable Homeomorphism in a Flow Subject to a Regularity Condition on the Derivatives of the Positive Transition Homeomorphisms*

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1. INTRODUCTION

We will consider differentiable homeomorphisms on a prescribed open interval X which may be unbounded. As in topological dynamics by a (continuous) flow on X we mean a triple (X, R, π), where R denotes the additive group of real numbers and $\pi: X \times R \to X$ is continuous and (X, R, π) satisfies

(i) $\pi(x, 0) = x, \qquad x \in X.$

(ii) $\pi(\pi(x, t)s) = \pi(x, t + s); \qquad x \in X, \quad s, t \in R$ (cf. [5]).

For each fixed $t \in R$ a transition homeomorphism $\pi^t: X \to X$, $x \mapsto \pi(x, t)$, is defined; (i) and (ii) are then equivalent to

(i)′ π^0 is the identity map of X.

(ii)′ $\pi^s \circ \pi^t = \pi^{s+t}; \qquad s, t \in R.$

We say that a (self-) homeomorphism f of X is embedded in (X, R, π) if $f = \pi^1$. In this paper f is assumed to be differentiable, has a differentiable inverse and has no fixed point; the case for f having a fixed point can also be considered (see [9]). We will define and study a class of differentiable homeomorphisms which we name as p-homeomorphism (the p does not refer to any point). We will study the embedding and the uniqueness of embedding of such homeomorphisms. The main result is given in Theorem 4.1, which, among other things, shows that the flow in which f is embedded can have the semigroup $T^+ = \{\pi^t \mid t > 0\}$ consisting of only p-homeomorphisms. If in addition f is C^1, this flow extended to the appropriate end-point, if finite, can be a C^1-flow; this result can be used to study embedding homeomorphisms in C^1-flows (see [9]). If f' and $(f^{-1})'$ are bounded on a compact subinterval of X and f'

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is of bounded variation near an appropriate end-point, then f can be shown to be a p-homeomorphism. We refer the readers to Theorems 4.2 and 4.1 for such results. Related results on embedding of homeomorphisms can be found in Coifman [3], Fort [4], and Sternberg [10].

We now define a p-homeomorphism. The four conditions in the following are needed for proving Theorem 4.1; with (i)–(iii) only, Theorem 4.1 does not hold even under the assumption that the extension of f to the end-point is C^1 (Example 4.4 of [8] shows that the principal Schöder function fails to define, then the transition homeomorphisms do not all have derivatives at the end-point.)

DEFINITION 1.1. Let f be a differentiable homeomorphism of an open interval X. We name f a *p-homeomorphism* if it satisfies the following properties.

(i) f has no fixed point.

(ii) f' exists and is positive.

(iii) $\rho_f = \lim_{x\to p_0} f'(x)$ exists, where $p_0 = \lim_{n\to\infty} f^n(x)$, $x \in X$ ($| p_0 | \leqslant \infty$).

(iv) $F(x, c) = \lim_{n\to\infty}(f^n)'(x)/(f^n)'(c)$ exists for each $c \in X$ and for every $x \in X$. For each compact set K in $X \times X$, $F(x, c)$ is bounded for $(x, c) \in K$ and the sequence $(f^n)'(x)/(f^n)'(c)$ converges uniformly.

We note that if (i), (ii), and (iii) hold, property (iv) of Definition 1.1 may be reformulated as follows. For a certain $c_0 \in X$, f has the property that if $[s_0, s]$ is a finite closed subinterval of X, then there exist constants $k > 0$ and $K > 0$ such that

$$k < \limsup\{(f^n)'(x)/(f^n)'(c_0) \mid x \in [s_0, s_1]\} < K$$

and for each $\epsilon > 0$ there exists an integer N such that

$$\sup\{|(f^n)'(x)/(f^n)'(c_0) - (f^m)'(x)/(f^m)'(c_0)| \quad x \in [s_0, s_1]\} < \epsilon$$

for $n, m > N$.

In the rest of the paper f will denote a p-homeomorphism.

2. ASSOCIATE FUNCTIONS OF f

We fix a c in (iv) of Definition 1.1. For fixed y the function $(f^n(x) - f^n(y))/(f^n)'(c)$ is monotone-increasing and differentiable, which is then absolutely continuous. By taking differentiation and (Lebesgue) integration we have

$$\int_y^x ((f^n)'(t)/(f^n)'(c))\, dt = (f^n(x) - f^n(y))/(f \ \ )'(c)$$

If $F(t, c)$ is bounded and if $(f^n)'(t)/(f^n)'(c)$ converges to $F(t, c)$ uniformly for $t \in [y, x] \subset X$, then

$$\int_y^x F(t, c)\, dt = \lim_{n\to\infty} \int_y^x (f^n)'(t)/(f^n)'(c)\, dt; \qquad x, y \in X$$

Then by the uniform convergence of Definition 1.1(iv), hence that the $(f^n(x) - f^n(y))/(f^n)'(c)$

$$\begin{aligned}(\partial/\partial x) \int_y^x F(t, c)\, dt &= (\partial/\partial x)(\lim_{n\to\infty}(f^n(x) - f^n(y))/(f^n)'(c))\\ &= \lim_{n\to\infty}(\partial/\partial x)((f^n(x) - f^n(y))/(f^n)'(c)\\ &= F(x, c)\end{aligned}$$

Note that $F(x, c) > 0$ by (iv) of Definition 1.1. We introduce a function γ for f by defining

$$\gamma(x) = \lim_{n\to\infty} \frac{(f^n)'(x)}{f^{n+1}(x) - f^n(x)} = F(x, c)\Big/\int_x^{f(x)} F(t, c)\, dt, \qquad x \in X.$$

Lemma 2.1. *The function γ is well-defined on X. $\gamma(x) > 0$, $x \in X$ if $x < f(x)$; $\gamma(x) < 0$, $x \in X$ if $x > f(x)$.*

Lemma 2.2. $\rho_f \neq 0$.

Proof. We have, by the mean value theorem,

$$\lim_{n\to\infty} \frac{f^{n+2}(x) - f^{n+1}(x)}{f^{n+1}(c) - f^n(c)} = \rho_f \cdot \lim_{n\to\infty} \frac{f^{n+1}(x) - f^n(x)}{f^{n+1}(c) - f^n(c)}$$

or

$$0 < \frac{\int_{f(x)}^{f^2(x)} F(t, c)\, dt}{\int_c^{f(c)} F(t, c)\, dt} = \rho_f \cdot \frac{\int_x^{f(x)} F(t, c)\, dt}{\int_c^{f(c)} F(t, c)\, dt}.$$

We define two functions, one for $\rho_f = 1$ and one for $\rho_f \neq 1$.

Definition 2.3. We define for f:

(a) $I_0(x) = \int_c^x \gamma(t)\, dt, \qquad x \in X\ (\rho_f = 1)$

(b) $I(x) = \int_c^x \frac{\rho_f - 1}{\log \rho_f}\, \gamma(t)\, dt,\ \rho_f \neq 1, \qquad x \in X$

THEOREM 2.4. *The following properties hold.*

(i) *If* $\rho_f = 1$, *then* $I_0(f(x)) = I_0(x) + 1, \qquad x \in X.$

(ii) *If* $\rho_f \neq 1$, *then* $I(f(x)) = I(x) + 1, \qquad x \in X.$

I_0 *and* I *are differentiable* (*strictly*) *monotone functions of* X *onto the set of real numbers, whose derivatives do not vanish.*

Proof. (i) By the mean value theorem

$$\int_{f(c)}^{f(x)} F(t, c)\, dt = \rho_f \cdot \int_c^x F(t, c)\, dt = \int_c^x F(t, c)\, dt.$$

Then

$$\int_c^{f(c)} F(t, c)\, dt = \int_x^{f(x)} F(t, c)\, dt, \qquad x \in X. \tag{1}$$

Hence

$$\begin{aligned} I_0(f(x)) - I_0(x) &= \int_c^{f(x)} F(t, c)\, dt \Big/ \int_c^{f(c)} F(t, c)\, dt \\ &\quad - \int_c^x F(t, c)\, dt \Big/ \int_c^{f(c)} F(t, c)\, dt = 1. \end{aligned}$$

(ii) Let $1 < \rho_f$ or $\rho_f < 1$. We have

$$\begin{aligned} F(f(x), c) &= \prod_{n=1}^{\infty} \frac{f'(f^n(f(x)))}{f'(f^n(c))} \\ &= \frac{\lim_{n\to\infty} f'(f^n(f(x)))}{f'(x)} \cdot \prod_{n=1}^{\infty} \frac{f'(f^n(x))}{f'(f^n(c))} \\ &= \frac{\rho_f}{f'(x)} \cdot F(x, c), \qquad x \in X. \end{aligned}$$

Then

$$\begin{aligned} I(f(x)) - I(x) &= \int_c^{f(x)} \frac{\rho_f - 1}{\log \rho_f} \gamma(t)\, dt - \int_c^x \frac{\rho_f - 1}{\log \rho_f} \gamma(t)\, dt \\ &= \int_x^{f(x)} \frac{\rho_f - 1}{\log \rho_f} \cdot \frac{F(t, c)}{\int_t^{f(t)} F(s, c)\, ds}\, dt \\ &= \frac{1}{\log \rho_f} \int_x^{f(x)} \frac{dv}{v}, \end{aligned}$$

where

$$v = \int_t^{f(t)} F(s, c)\, ds.$$

Hence

$$I(f(x)) - I(x) = \frac{1}{\log \rho_f} \left[\log \int_{f(x)}^{f^2(x)} F(t, c)\, dt - \log \int_x^{f(x)} F(t, c)\, dt\right] \tag{2}$$

$$= \frac{1}{\log \rho_f} \left[\log \left(\rho_f \int_x^{f(x)} F(t, c)\, dt\right) - \log \int_x^{f(x)} F(t, c)\, dt\right]$$

$$= 1.$$

The functions I_0 and I are Abel functions for f.

3. Behavior of f Near Stable End-Point p_0

In this section we obtain a condition on uniform convergence for f, to be used for proving the main theorem (next section).

LEMMA 3.1. *For each finite closed subinterval* $[s_0, s_1]$ *of* X *and for each* $\epsilon > 0$ *there exists an integer* N *for which*

$$\sup\{| F(f^n(x), f^n(y)) - 1 | \mid x, y \in [s_0, s_1]\} < \epsilon \qquad \textit{if} \quad n > N.$$

Proof. We have

$$F\left(f^n(x), f^n(y) = \prod_{i=0}^{\infty} f'(f^{n+i}(x))/f'(f^{n+i}(y)\right)$$

$$= ((f^n)'(y)/(f^n)'(x))\Big/\left(\prod_{i=0}^{\infty} f'(f^i(y))/f'(f^i(x))\right)$$

$$= ((f^n)(y)/(f^n)'(x)/F(y, x) \tag{3}$$

It follows from

$$((f^n)'(y)/(f^n)'(x))/F(y, x) = [((f^n)'(y)/(f^n)'(x) - F(y, x))\, F(x, c_0)/F(y, c_0)] + 1$$

and the remark in the closing of Section 1, that for k, K given there that

$$1 - K(\epsilon_n/k) < ((f^n)'(y)/(f^n)'(x))/F(y, x) < 1 + K(\epsilon_n/k) \tag{4}$$

where $\epsilon_n = \sup\{|(f^n)'(y)/(f^n)'(x) - F(y, x)| \mid x, y \in [s_0, s_1]\}$. Since $\lim_{n\to\infty} \epsilon_n = 0$, the lemma follows from (3) and (4).

LEMMA 3.2. *Let* S *be a set of functions on* X *such that*

(i) $f(g(x)) = g(f(x)); \qquad g \in S, \quad x \in X$

and

(ii) *If* $[a_1, b_1]$ *is a finite closed subinterval of* $X = (a, b)$, *then*

$$a < \inf\{g(x) \mid g \in S,\ x \in [a_1, b_{,1}]\} \leqslant \sup\{g(x) \mid g \in S,\ x \in [a_1, b_{,1}]\} < b.$$

Then

$$\lim_{x \to p_0} F(g(x), x) = 1 \text{ uniformly (independent of } g \in S).$$

Proof. Without loss of generality we may assume $x < f(x)$, $x \in X$. For each $y \in X$, there exists some $x = x(y)$ in $[c, f(c)]$ and an integer $n = n(y)$ such that $y = f^n(x)$. Then by (i) for $g \in S$

$$F(g(y), y) = F(g(f^n(x)), f^n(x)) = F(f^n(g(x)), f^n(x)). \tag{5}$$

By (ii) there exists a finite closed subinterval $[s_0, s_1]$ in X such that

$$s_0 < \min\{c, \inf\{g(x) \mid g \in S,\ x \in [c, f(c)]\} \tag{6a}$$

and

$$s_1 > \max\{f(c), \sup\{g(x) \mid g \in S,\ x \in [c, f(c)]\}. \tag{6b}$$

It follows from the preceding lemma that there exists an integer N such that

$$\sup\{|F(f^n(x), f^n(x')) - 1| \mid x, x' \in [s_0, s_1]\} < \epsilon \qquad \text{if} \quad n > N.$$

By (5) and (6)

$$|F(g(y), y) - 1| < \epsilon \qquad \text{if} \quad f^N(f(c)) < y < b.$$

The proof of Lemma 3.2 is completed.

4. Embedding f in a Flow

The results obtained are now used to obtain the following embedding theorem. For an arbitrary differentiable function g: $X \to X$, $g' > 0$, we define

$$F^g(x, y) = \lim_{n \to \infty} (g^n)'(x)/(g^n)'(y)$$

if the limit exists.

THEOREM 4.1. *Let f be a p-homeomorphism. Then the functional I_0 (for the case $\rho_f = 1$) or I (for the case $\rho_f \neq 1$) induces a continuous flow (X, R, π) for which the following properties hold.*

(a) *$f^n(x) = \pi^n(x)$, n integral; i.e., f is embedded in the flow.*

(b) *Each π^t $(t > 0)$ is a p-homeomorphism, hence each π^t is differentiable and $\lim_{x \to p_0} (\pi^t)'(x) > 0$ exists.*

(c) *For* $g = \pi^t$ $(t > 0)$ *the sequence of functions* $\{(g^n)'(x)/(g^n)(c)\}$ *converges in* X *and converges uniformly for* $(x, t) \in J \times \Lambda$, *where* J *is a finite closed subinterval of* X *and* Λ *is a finite closed subinterval of* $(0, \infty) \subset R$.

(d) $F^g(x, y) = F(x, y), \quad x, y \in X, \quad g = \pi^t, \quad t > 0.$

(e) *If* $f \in C^1$ *then* $\pi^t \in C^1$ *for each* $t \in R$. *If in addition the point* p_0 *is finite, then the extension of* π^t *to the fixed point* p_0 *is of class* C^1.

Proof. We first define the flow (X, R, π). According to Theorem 2.4 the following flow (X, R, π) can be defined by requiring

$$A(\pi^t(x)) = A(x) + t, \qquad x \in X, \quad t \in R, \tag{7}$$

where A in Eq. (7) is the function I_0 if $\rho_f = 1$ and the function I if $\rho_f \neq 1$. It is readily seen that (a) of the theorem is true.

We differentiate (7) to obtain

$$\begin{aligned}(\pi^t)'(x) &= \frac{\gamma(x)}{\gamma(\pi^t(x))} \\ &= F(x, \pi^t(x)) \int_{\pi^t(x)}^{f(\pi^t(x))} F(t, c)\, dt \Big/ \int_x^{f(x)} F(t, c)\, dt.\end{aligned} \tag{8}$$

This is reduced to the equation

$$(\pi^t)'(x) = \rho_f{}^t \cdot F(x, \pi^t(x)), \qquad x \in X, \quad t \in R \tag{9}$$

Thus, for the case $\rho_f = 1$ Eq. (1) holds, hence (9) is true. For the cases $\rho_f < 1$ and $\rho_f > 1$ we let f on the left-hand side of (2) be the map π^t and we note that the function A in (7) is the function I. We then have

$$t = I(\pi^t(x)) - I(x) = (1/\log \rho_f) \left[\log \left(\int_{\pi^t(x)}^{f(\pi^t(x))} F(t, c)\, dt \Big/ \int_x^{f(x)} F(t, c)\, dt\right)\right].$$

It follows that

$$\int_{\pi^t(x)}^{f(\pi^t(x))} F(t, c)\, dt = \rho_f{}^t \cdot \int_x^{f(x)} F(t, c)\, dt$$

and Eq. (9) follows for these cases.

Now for proving (b) we have to show that each π^t $(t > 0)$ of (7) is a p-homeomorphism. Following that f itself has no fixed point, each π^t has no fixed point unless $t = 0$. Now $\lim_{x \to p_0} F(\pi^t(x), x) = 1$ (see Lemma 3.2), by (9)

$$\lim_{x \to p_0} (^t)'(x) = \rho_f{}^t > 0, \qquad t \in R. \tag{10}$$

Thus we can prove (b) if we can prove (c).

For proving (c) and (d) we let $g = \pi^t$, $t > 0$, $t \in F$. By (9)

$$(g^n)'(x)/(g^n)'(c) = F(x, g^n(x))/F(c, g^n(c)) = F(x, c) \cdot F(g^n(c), g^n(x)). \tag{11}$$

Let $\Delta = \{s \in R \mid \pi^s(c) \in J\}$, then Δ is a finite closed subinterval of R. By the definition of f if

$$M = \sup\{F(\pi^s(c), c) \mid s \in \Delta\}, \qquad \text{then} \quad M < \infty.$$

If $x \in J$, $t \in \Lambda$, there exists a unique $s \in \Delta$ such that $\pi^s(c) = x$. Then by (11)

$$(g^n)'(x)/(g^n)'(c) - (g^m)'(x)/(g^m)'(c) = F(\pi^s(c), c)(F(\pi^{-s}(y_n), y_n) - F(\pi^{-s}(y_m), y_m)),$$
$$|(g^n)'(x)/(g^n)'(c) - (g^m)'(x)/(g^m)'(c)| \leqslant M \mid F(\pi^{-s}(y_n), y_n) - F(\pi^{-s}(y_m), y_m)|, \tag{12}$$

where $y_n = g^n(x) = \pi^{nt}(x)$.

It is not difficult to see that $\{y_n\}$ in (12) satisfies

$$\lim_{n\to\infty} y_n = p_0 \text{ uniformly (independent of } x \in J \text{ and } t \in \Lambda).$$

It follows that

$$\lim_{n\to\infty} F(\pi^{-s}(y_n), y_n) = 1 \text{ uniformly (independent of } (x, t) \in J \times \Lambda). \tag{13}$$

(Use Lemma 3.2 by letting $S = \{\pi^{-s} \mid s \in \Delta\}$.) Then the proofs for (c) and (d) follow; (c) follows from (12) and (d) follows from (11), by applying (13). (e) follows from the limit in (b) and that $F(x, \pi^t(x))$ in (9) is now continuous.

The proof of the theorem is completed.

We give a sufficient condition for p-homeomorphisms.

THEOREM 4.2. *If X is an arbitrary open interval and h is a self-homeomorphism of X such that* (i) *h has no fixed point,* (ii) *h' exists >0,* (iii) *h' and $(h^{-1})'$ are bounded on compact subintervals of X, and* (iv) *there exists a neighborhood of p_0 in which h' is of bounded variation, then h is a p-homeomorphism.*

Proof. The proof is a straight computation and is omitted.

5. Functions Commuting with f and Uniqueness of Embedding

Case $\rho_f \neq 1$. We consider the principal Schröder function [8]

$$L(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{f^n(x) - p_0}{f^n(c) - p_0}, & |p_0| < \infty \\ \lim\limits_{n\to\infty} f^n(x)/f^n(c), & |p_0| = \infty \end{cases}$$

It was proved in [8] that in case the directional limit κ_f exists $\neq 0, 1$, and in case f can be embedded in an E_{p_0}-flow, ([8], p. 278, Definition 2.2) then the flow must be given by $\pi^t(x) = L^{-1}(\kappa_f{}^t L(x))$ (see Theorem 2.5 of [8]). Since the flow in Theorem 4.1 is an E_{p_0}-flow we obtain the uniqueness of embedding of Theorem 4.1 for $\rho_f = \kappa_f \neq 1$. In fact the uniqueness of embedding can be considered a special property of uniqueness of functions commuting with f. In terms of commuting functions we have the following more general theorem.

THEOREM 5.1. *If f is a p-homeomorphism and $\rho_f \neq 1$, then f can be embedded in a flow of Theorem* 4.1 *which is the only E_{p_0}-flow in which f can be embedded. The family of mappings g: $X \to X$ which has a directional derivative and which commutes with f is the set of transition homeomorphisms of the flow.*

Proof. We apply Theorem 2.6 of [8]. The only thing need to be verified is that $\kappa_f \neq 0, 1$. This is true if we can show $\rho_f = \kappa_f$. The proof for it is quite direct and is omitted.

Case $\rho_f = 1$. For this case $L \equiv 1$ (the uniqueness of Theorem 5.1 fails).

DEFINITION 5.2. Let X be an arbitrary interval, p_0 an end-point of X, $|p_0| \leqslant \infty$, (X, R, π) a flow on X. The flow is said to be an *S_{p_0}-flow* if each $(\pi^t)'$ exists and each $\lim_{x \to p_0} (\pi^t)'(x)$ exists.

THEOREM 5.3. *If f is a p-homeomorphism and $\rho_f = 1$, then f can be embedded in a flow of Theorem* 4.1 *which is the only S_{p_0}-flow in which f can be embedded. The family of mappings g: $X \to X$ which is differentiable near p_0, for which $\lim_{x \to p_0} g'(x)$ exists and which commutes with f is the set of transition homeomorphisms of the flow. Let G denote the family of mappings g: $X \to X$ which is differentiable near p_0 and for which $\lim_{x \to p_0} g'(x)$ exists, let $G_{g_1} = \{g \in G \mid g \circ g_1 = g_1 \circ g\}$, $g_1 \in G$. Then $G_h = G_f = \{\pi^t \mid t \in R\}$ for $h \in G_f$ which is not the identity mapping.*

Proof. We prove the statement for g. According to the last statement of [8, Theorem 2.5] we have $L(\pi^t(c)) = \kappa_f{}^t = 1$ for all $t \in R$. Then $L \equiv 1$ and if $|p_0| < \infty$ then

$$\rho_g = \kappa_g = \lim_{n\to\infty} \frac{g(f^n(c)) - p_0}{f^n(c) - p_0} = \lim_{n\to\infty} \frac{f^n(g(c)) - p_0}{f^n(c) - p_0} = L(g(c)) = 1.$$

Likewise if $|p_0| = \infty$ then $\rho_g = \kappa_g = 1$. Similar to the proof of (1) we can show

$$\int_x^{g(x)} F(t, c)\, dt = \int_c^{g(c)} F(t, c)\, dt, \qquad x \in X.$$

Now if $x \in X$, $g(x) = \pi^t(x)$ for some $t \in R$. By (7)

$$\begin{aligned} t &= I_0(\pi^t(x)) - I_0(x) = I_0(g(x) - I_0(x) \\ &= \int_x^{g(x)} F(t, c)\, dt \Big/ \int_c^{f(c)} F(t, c)\, dt \\ &= \int_c^{g(c)} F(t, c)\, dt \Big/ \int_c^{f(c)} F(t, c)\, dt. \end{aligned}$$

Hence t is independent of x. Then $g = \pi^t$, $t = I_0(g(c))$. Therefore the family of mappings which is differentiable near p_0 for which the limit of derivative as x approaches p_0 exists and which commutes with f is the family of transition homeomorphisms of the flow (X, R, π).

For any S_{p_0}-flow (X, R, σ) in which π^τ $(\tau \neq 0)$ is embedded the relation $t = I_0(g(c))$, where I_0 is now the function for π^τ, establishes a linear relation $\sigma^s = \pi^{t'}$, between the parameter s of the S_{p_0}-flow with t' which is a scalar multiple of the parameter t in (X, R, π). The flow (X, R, σ) then differs by (X, R, π) by at most a linear reparametrization. In particular, for $\tau = 1$ we have $s = t' = t$ and the S_{p_0}-flow is then the same as (X, R, π).

The proof of Theorem 5.3 is completed.

References

1. N. Abel, "Works," Vol. 2, Posthumous Paper (1881), pp. 36–39.
2. U. T. Bödewadt, Zur Iteration reeller Funktionen, *Math. Z.* **49** (1944), 497–516.
3. R. Coifman, Sur l'unicité des solutions de l'équation d'Abel–Schröder et l'itération continue, *J. Austral. Math. Soc.* **5** (1965), 36–47.
4. M. K. Fort, Jr., The embedding of homeomorphisms in flows, *Proc. Amer. Math. Soc.* **6** (1955), 960–967.
5. W. Gottschalk and G. A. Hedlund, "Topological Dynamics," Amer. Math. Soc., Colloq. Publ. 36, Providence, R.I. 1955.
6. H. Kneser, Reelle analytische Lösungen der Gleichung $\phi\{\phi(x)\} = e^x$ und verwandter Funktionalgleichungen, *J. Reine Angew. Math.* **187** (1949), 56–57.
7. M. Kuczma, A remark on commutable functions and continuous iterations, *Proc. Amer. Math. Soc.* **13** (1962), 847–850.
8. P.-F. Lam, Embedding homeomorphisms in differential flows, *Colloq. Math.* **35** (1976), 275–287.
9. P.-F. Lam, Embedding a homeomorphism in a flow subject to differentiability conditions, *in* "Topological Dynamics: An International Symposium (Colorado State University)," pp. 319–333, Benjamin, New York, 1968.
10. S. Sternberg, Local C^n transformations of the real line, *Duke Math. J.* **24** (1957), 97–102.